We see that the coefficient of x^{n-2} for $p(x)$ is

$$\frac{-s_1}{(a_1-a_2)(a_1-a_3)\cdots(a_1-a_n)}+\frac{-s_2}{(a_2-a_1)(a_2-a_3)\cdots(a_2-a_n)}+\cdots$$

$$+\quad\frac{-s_n}{(a_n-a_1)(a_n-a_2)\cdots(a_n-a_{n-1})}=0.$$

The left side of the last equality is equal to or the negative of the summation in the problem. Thus the summation in the problem is zero.

Also solved by the proposer.

- **5247:** *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania*

Calculate

$$\lim_{n\to\infty}\frac{1}{n}\sqrt[n]{\int_0^1 \ln(1+e^x)\ln(1+e^{2x})\cdots\ln(1+e^{nx})\ dx}.$$

Solution 1 by Anastasios Konronis, Athens, Greece

For $n\in N,\ x\in(0,1]$ we have

$$\begin{aligned}\ln(1+e^x)\cdot\ln(1+e^{2x})\cdots\ln(1+e^{nx}) &= n!x^n\prod_{k=1}^n\left(1+\frac{\ln(1+e^{-kx})}{kx}\right)=n!x^n\prod_{k=1}^n\left(1+\mathcal{O}\left(\frac{e^{-kx}}{kx}\right)\right)\\ &= n!x^n\left(1+\mathcal{O}\left(\frac{e^{-x}}{x^n}\right)\right)\\ &= n!\left(x^n+\mathcal{O}\left(e^{-x}\right)\right)\end{aligned}$$

so

$$\int_0^1 \ln(1+e^x)\cdot\ln(1+e^{2x})\cdots\ln(1+e^{nx})=\frac{n!}{n+1}\left(1+\mathcal{O}(n)\right).$$

Now from the above and taking into account that, from Stirling's formula,

$$\ln n!=n\ln n-n+\mathcal{O}(\ln n)$$

we get that

$$\begin{aligned}\frac{1}{n}\sqrt[n]{\int_0^1 \ln(1+e^x)\cdot\ln(1+e^{2x})\cdots\ln(1+e^{nx})\,dx} &= \frac{1}{n}\exp\left(\frac{1}{n}\ln\left(\frac{n!}{n+1}\left(1+\mathcal{O}(n)\right)\right)\right)\\ &= \frac{1}{n}\exp\left(\ln n-1+\mathcal{O}\left(\frac{\ln n}{n}\right)\right)=e^{-1}+\mathcal{O}\left(\frac{\ln n}{n}\right)\to e^{-1}\end{aligned}$$

Solution 2 by Arkady Alt, San Jose, California, USA.

Let $f_n(x) = \prod_{k=1}^{n} \ln\left(1+e^{kx}\right)$. Since $f_n(x) > \prod_{k=1}^{n} \ln\left(e^{kx}\right) = x^n n!$ then

$$\int_0^1 f_n(x)\,dx > n! \int_0^1 x^n dx = \frac{n!}{n+1}.$$

On the other hand, since $f_n(x) f_n(1) \le 1$ we have

$$\int_0^1 f_n(x)\,dx \le f_n(1) \int_0^1 dx = f_n(1).$$

Thus,

$$\frac{1}{n}\sqrt[n]{\frac{n!}{n+1}} < \frac{1}{n}\sqrt[n]{\int_0^1 f_n(x)\,dx} \le \frac{1}{n}\sqrt[n]{f_n(1)}.$$

Let $a_n = \dfrac{f_n(1)}{n^n}$.
Since

$$\begin{aligned}
\lim_{n\to\infty} \frac{a_n}{a_{n-1}} &= \lim_{n\to\infty} \left(\frac{f_n(1)}{n^n} \cdot \frac{(n-1)^{n-1}}{f_{n-1}(1)}\right) \\
&= \lim_{n\to\infty} \left(\left(1-\frac{1}{n}\right)^{n-1} \cdot \frac{\ln(1+e^n)}{n}\right) \\
&= \lim_{n\to\infty} \left(1-\frac{1}{n}\right)^{n-1} \cdot \lim_{n\to\infty} \frac{\ln(1+e^{-n})+n}{n} \\
&= e^{-1}\cdot 1 = e^{-1}
\end{aligned}$$

then by *, the Multiplicative Stolz Theorem $\lim_{n\to\infty} \frac{1}{n}\sqrt[n]{f_n(1)} = \lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{a_n}{a_{n-1}} = e^{-1}$.

Also we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n}\sqrt[n]{\frac{n!}{n+1}} &= \lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} \cdot \frac{1}{\sqrt[n]{n+1}} \\
&= \lim_{n\to\infty} \sqrt[n]{n!}n \cdot \lim_{n\to\infty} \frac{1}{\sqrt[n]{n+1}} \\
&= e^{-1}\cdot 1 = e^{-1}.
\end{aligned}$$

(Note: $\lim_{n\to\infty} \dfrac{\sqrt[n]{n!}}{n} = e^{-1}$. Indeed,

$$\left(\frac{n}{e}\right)^n < n! < \left(\frac{n+1}{e}\right)^n (n+1) \Rightarrow$$

$$\frac{1}{e} < \frac{\sqrt[n]{n!}}{n} < \frac{1}{e}\cdot\frac{n+1}{n}\cdot\sqrt[n]{n+1},$$

or again, applying the Multiplicative Stolz Theorem to $\sqrt[n]{\frac{n!}{n^n}}$).

Then by the squeeze principle,

$$\lim_{n\to\infty}\frac{1}{n}\sqrt[n]{f_n(1)} = \lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n} = e^{-1} = e^{-1} \text{ yields}$$

$$\lim_{n\to\infty}\frac{1}{n}\sqrt[n]{\int_0^1 f_n(x)\,dx} = e^{-1}.$$

* We use the Multiplicative Stolz Theorem in the following form:
If the sequence $\left(\frac{a_{n+1}}{a_n}\right)_{n\geq 1}$ has a limit then the sequence $\left(\sqrt[n]{a_n}\right)_{n\geq 1}$ has a limit and

$$\lim_{n\to\infty}\sqrt[n]{a_n} = \lim_{n\to\infty}\frac{a_{n+1}}{a_n}.$$

Solution 3 by Kee-Wai, Hong Kong, China

We show that the limit equals $\frac{1}{e}$.

Denote the integrand by $f(x)$. Since $f(x) > (x)(2x)\cdots(nx) = (n!)x^n$, so

$$\int_0^1 f(x)dx > \frac{n!}{n+1}. \qquad (1)$$

For $0 \leq x \leq 1$ and $k = 1, 2, \cdots, n$, we have

$$1 + e^{kx} \leq 1 + e^k < 2e^k < e^{1+k}, \text{so that}$$

$$f(x) < (n+1)! \text{ and}$$

$$\int_0^1 f(x)dx < (n+1)!. \qquad (2)$$

By Stirling's formula for $n!$ we have

$$\lim_{n\to\infty}\frac{1}{n}\sqrt[n]{\frac{n!}{n+1}} = \lim_{n\to\infty}\frac{1}{n}\sqrt[n]{(n+1)!} = \frac{1}{e}.$$

Now by $(1), (2)$ and the squeezing principle, we obtain the result we claimed.

Also solved by Paul M. Harms, North Newton, KS; Adrian Naco, Polytechnic University, Tirana, Albania and the proposer.

Mea Culpa (yet again)

Featured solution 5241(3) that appeared in the April 2013 issue of the column was submitted jointly by **Anastasios Kotronis and Konstantinos Tsouvalas, University of Athens, Athens, Greece.** I inadvertently forgot to list Konstantinos' name. Sorry.